Some notes concerning graceful digraphs on up to 6 nonisolated vertices

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Abstract

We try to answer exercise 180 posed in [1]. This exercise asks to investigate the graceful digraphs that have at most 6 nonisolated vertices.

1 Introduction

Assume we have a directed graph with n vertices and m arcs. We can assign to the vertices of this digraph distinct integer labels $l(v)$, with $0 \leq l(v) \leq m$. Each arc $v \longrightarrow w$ then receives the label $(l(w) - l(v)) \bmod (m + 1)$. The digraph is graceful if there is exactly one arc labeled k, for $1 \leq k \leq m$.

We investigate the gracefulness of all digraphs having up to 6 nonisolated vertices.

2 Methodology

The 1540944 digraphs with up to 6 nonisolated vertices can be downloaded from [3]. And to get all graceful labelings of digraphs, given the number of vertices and number of edges, a program is available at [2].

The main thing we need to do is determine a unique 'signature' for each of these digraphs. When two digraphs have a different signature, they cannot be isomorphic. The program [2] already calculates a 32-bit signature. The signature calculation depends on a random seed value. When there are not too many digraphs, a unique signature can be found by using the proper seed. But this does not give a unique signature for the 1540944 digraphs we want to investigate. Donald Knuth suggested to use a 64-bit signature instead, a combination of 2 32-bit signatures with different seeds. This larger signature indeed is unique for all of our digraphs. Using this unique signature, it is possible to determine which labelings correspond to each digraph.

The result of these calculations can be found at [4]. There is a file 'diadj6hash', which contains the hashes for all the digraphs. The file 'hashesandlabelings' contains the hash that corresponds to each labeling. The labelings

are encoded using alphanumeric characters, $a = 0, b = 1, \ldots$ For example, the digraph with labeling "abacadba" has edges $0 \longrightarrow 1$, $0 \longrightarrow 2$, $0 \longrightarrow 3$, $1 \longrightarrow 2$.

3 Results

Number of graceful digraphs Not all digraphs have a graceful labeling. Some of them do not have enough edges. And some can't be gracefully labeled, even when there are enough edges. Here are the statistics, by the number of digraphs on at most n nonisolated vertices:

Up to n	Graceful	Ungraceful too few edges	Ungraceful other	Total
1	1	0	0	1
2	3	0	0	3
3	13	0	3	16
4	152	1	65	218
5	5882	4	3722	9608
6	844164	20	696760	1540944

Acyclic digraphs If we only take into account acyclic digraphs, we get the following counts:

Up to n	Graceful	Ungraceful too few edges	Ungraceful other	Total
1	1	0	0	1
2	2	0	0	2
3	5	0	1	6
4	25	1	5	31
5	266	4	32	302
6	5828	19	137	5984

Dependence on number of edges and vertices The next table shows how many graphs with n vertices and m edges are graceful, and how many there are in total.

			n			
m	1	2	3	4	5	6
0	1/1	0/0	0/0	0/0	0/0	0/0
1		0/1	0/0	0/0	0/0	0/0
2		1/1	2/3	0/1	0/0	0/0
4			3/4	16/23	17/34	0/15
5			1/1	32/37	103/116	113/134
6			1/1	29/47	241/331	565/664
7				30/38	601/669	2477/2535
8				15/27	788/1128	6982/7796
9				6/13	1109/1477	18446/19719
10				3/5	868/1665	34661/42193
11				1/1	988/1489	71920/77324
12				1/1	466/1154	87664/122960
13					321/707	130654/170317
14					141/379	127694/206983
15					55/154	132728/220768
16					20/61	75241/207301
17					8/16	78932/171008
18					3/5	30734/124110
19					0/1	23772/78813
20					1/1	9316/43862
21						4114/21209
22						1278/8951
23						748/3242
24						165/1043
25						56/288
26						18/76
27						3/17
28						0/5
29						0/1
30						1/1

Uniquely graceful digraphs The digraphs that have a unique labeling are especially interesting. Let's first check how many there are in total, and how many are weakly and strongly connected:

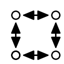
Up to n	Total	Weakly connected	Strongly connected
1	1	1	1
2	3	3	2
3	11	11	4
4	70	69	16
5	1500	1494	694
6	216467	216416	150328

If we zoom in on the unique labelings of strongly connected digraphs, we can count how many of them have n vertices and m edges:

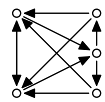
m	n = 1	2	3	4	5	6
0	1	0	0	0	0	0
1		0	0	0	0	0
2		1	0	0	0	0
3			0	0	0	0
4			1	0	0	0
5			0	0	0	0
6			1	0	0	0
7				6	2	0
8				3	10	0
9				0	48	16
10				2	134	356
11				0	180	1018
12				1	153	6648
13					78	9692
14					50	21734
15					10	31913
16					10	26476
17					0	21786
18					2	14316
19					0	9950
20					1	3874
21						978
22						594
23						214
24						58
25						0
26						8
27						2
28						0
29						0
30						1

For each n, the strongly connected digraph with the most edges is $K_n^{\leftrightarrow}$. This is as expected, since we know that K_n is rainbow graceful for $n < 7$ ([1], exercise 193).

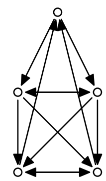
Using the nauty program [5], we can compute the number of symmetries of each of our digraphs. Here are some of the uniquely graceful strongly connected ones with a lot of symmetries:



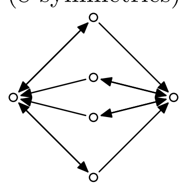
(8 symmetries)



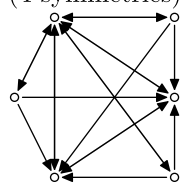
(4 symmetries)



(4 symmetries)



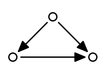
(8 symmetries)



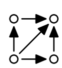
(12 symmetries)

For the weakly connected digraphs with many symmetries, we find mostly the k-partite digraphs. This is no surprise, since we know from [1], exercise 145 that all k-partite graphs are graceful.

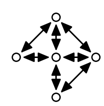
Most graceful digraphs This are the digraphs with the most graceful labelings for $3 \leq n \leq 6$:



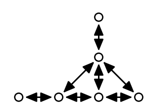
(2 labelings)



(7 labelings)



(32 labelings)

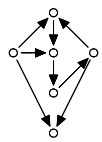


(540 labelings)

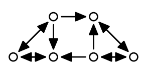
R-regular digraphs It is also interesting to check how many of our digraphs are r-regular. The following table shows for each n 'the number of graceful r-regular digraphs'/'total number of r-regular digraphs'.

			n			
r	1	2	3	4	5	6
1	0	1/1	0	0/1	0	0/1
2	0	1/1	1/2	3/5	0/6	9/17
3	0	0	0	4/6	0	114/119
4	0	0	1/1	3/5	19/40	226/494
5	0	0	0	0/1	0	472/774
6	0	0	0	1/1	2/6	57/494
7	0	0	0	0	0	0/119
8	0	0	0	0	1/1	3/17
9	0	0	0	0	0	0/1
10	0	0	0	0	0	1/1

The most graceful of the cubic graphs (14 labelings) is:



And the most graceful of all the regular digraphs is this 4-regular one (20 labelings):



Behaviour of random graceful digraphs with many edges As discussed in [1], we can construct random graceful digraphs with m edges by generating a random LO-array of size m. We can take random samples and check the properties of these random digraphs.

The obvious thing to look at first is how many vertices the random digraphs have for a given m. Here are the average number of vertices for various m, after taking 1000 random samples for each m: (10: 9.413, 100: 87.328, 1000: 865.546, 10000: 8646.162, 100000: 86467.81). According to Donald Knuth, it can be shown that $n \to (1 - e^{-2})m$ as $m \to \infty$.

Let's also check how many of the random digraphs are weakly connected. Here are the fraction of weakly connected digraphs for various m, after taking 10000 random samples for each m: (10: 0.71082, 20: 0.51303, 30: 0.38284, 40: 0.28768, 50: 0.21644, 60: 0.16434, 70: 0.12194, 80: 0.09052, 90: 0.06748). Weak connectedness occurs less and less as m increases.

We can also generate random digraphs for a given m and search for those with specific properties. For example, on the next page are the edges of the largest weakly connected digraph found using this method, having $n = 343$ nodes and $m = 410$ edges.

1	235-236	2	211-213	3	88-91	4	365-369	5	34-39	6	352-358	7	13-20
8	338-346	9	331-340	10	301-311	11	6-17	12	208-220	13	295-308	14	118-132
15	126-141	16	258-274	17	339-356	18	96-114	19	147-166	20	386-406	21	77-98
22	93-115	23	91-114	24	264-288	25	294-319	26	132-158	27	271-298	28	270-298
29	376-405	30	388-7	31	404-24	32	290-322	33	49-82	34	256-290	35	55-90
36	26-62	37	189-226	38	226-264	39	356-395	40	149-189	41	75-116	42	161-203
43	156-199	44	82-126	45	84-129	46	212-258	47	268-315	48	385-22	49	232-281
50	104-154	51	223-274	52	151-203	53	249-302	54	217-271	55	36-91	56	129-185
57	45-102	58	23-81	59	129-188	60	142-202	61	175-236	62	392-43	63	275-338
64	180-244	65	217-282	66	131-197	67	223-290	68	43-111	69	310-379	70	360-19
71	141-212	72	197-269	73	406-68	74	42-116	75	151-226	76	297-373	77	143-220
78	215-293	79	213-292	80	251-331	81	362-32	82	90-172	83	277-360	84	223-307
85	189-274	86	340-15	87	169-256	88	32-120	89	201-290	90	292-382	91	99-190
92	198-290	93	338-20	94	396-79	95	313-408	96	148-244	97	13-110	98	187-285
99	337-25	100	153-253	101	361-51	102	375-66	103	80-183	104	357-50	105	303-408
106	41-147	107	94-201	108	19-127	109	150-259	110	132-242	111	376-76	112	272-384
113	247-360	114	300-3	115	357-61	116	21-137	117	309-15	118	248-366	119	213-332
120	136-256	121	8-129	122	223-345	123	329-41	124	218-342	125	295-9	126	276-402
127	322-38	128	135-263	129	373-91	130	389-108	131	266-397	132	379-100	133	195-328
134	301-24	135	349-73	136	33-169	137	309-35	138	97-235	139	381-109	140	86-226
141	91-232	142	60-202	143	166-309	144	352-85	145	191-336	146	154-300	147	385-121
148	408-145	149	137-286	150	236-386	151	224-375	152	275-16	153	286-28	154	180-334
155	393-137	156	219-375	157	343-89	158	77-235	159	217-376	160	281-30	161	376-126
162	150-312	163	210-373	164	42-206	165	380-134	166	387-142	167	197-364	168	383-140
169	123-292	170	185-355	171	68-239	172	224-396	173	231-404	174	11-185	175	306-70
176	348-113	177	114-291	178	136-314	179	119-298	180	239-8	181	339-109	182	372-143
183	297-69	184	49-233	185	238-12	186	205-391	187	181-368	188	57-245	189	103-292
190	68-258	191	242-22	192	187-379	193	29-222	194	7-201	195	137-332	196	324-109
197	406-192	198	352-139	199	157-356	200	92-292	201	365-155	202	241-32	203	349-141
204	65-269	205	43-248	206	306-101	207	240-36	208	12-220	209	167-376	210	71-281
211	381-181	212	358-159	213	336-138	214	191-405	215	340-144	216	244-49	217	398-204
218	205-12	219	357-165	220	59-279	221	235-45	222	268-79	223	141-364	224	151-375
225	320-134	226	333-148	227	345-161	228	256-73	229	346-164	230	34-264	231	192-12
232	208-29	233	181-3	234	175-409	235	399-223	236	293-118	237	322-148	238	128-366
239	189-17	240	62-302	241	24-265	242	212-43	243	303-135	244	59-303	245	324-158
246	143-389	247	298-134	248	357-194	249	395-233	250	98-348	251	141-392	252	80-332
253	45-298	254	241-84	255	272-116	256	100-356	257	283-129	258	346-193	259	192-40
260	230-79	261	48-309	262	46-308	263	333-185	264	115-379	265	96-361	266	270-125
267	293-149	268	246-103	269	91-360	270	126-396	271	280-140	272	296-157	273	275-137
274	339-202	275	29-304	276	95-371	277	273-139	278	149-16	279	57-336	280	363-232
281	149-19	282	128-410	283	120-403	284	303-176	285	158-32	286	124-410	287	61-348
288	109-397	289	18-307	290	229-108	291	264-144	292	181-62	293	134-16	294	281-164
295	318-202	296	290-175	297	101-398	298	236-123	299	38-337	300	151-40	301	356-246
302	80-382	303	62-365	304	207-100	305	159-53	306	240-135	307	287-183	308	362-259
309	389-287	310	310-209	311	66-377	312	212-113	313	222-124	314	253-156	315	64-379
316	215-120	317	138-44	318	116-23	319	347-255	320	391-300	321	54-375	322	331-242
323	272-184	324	12-336	325	276-190	326	171-86	327	343-259	328	97-14	329	69-398
330	141-60	331	112-32	332	365-286	333	133-55	334	246-169	335	218-142	336	388-313
337	256-182	338	222-149	339	35-374	340	303-232	341	269-199	342	328-259	343	334-266
344	245-178	345	339-273	346	55-401	347	55-402	348	73-10	349	145-83	350	337-276
351	103-43	352	315-256	353	129-71	354	201-144	355	16-371	356	165-110	357	106-52
358	176-123	359	201-149	360	397-346	361	336-286	362	36-398	363	60-12	364	65-18
365	256-210	366	404-359	367	301-257	368	26-394	369	88-46	370	306-265	371	101-61
372	73-34	373	331-293	374	311-274	375	291-255	376	250-215	377	146-112	378	176-143
379	336-304	380	303-272	381	9-390	382	202-173	383	245-217	384	201-174	385	320-294
386	128-103	387	94-70	388	172-149	389	126-104	390	396-375	391	244-224	392	272-253
393	6-399	394	5-399	395	52-36	396	211-196	397	33-19	398	372-359	399	81-69
400	355-344	401	149-139	402	199-190	403	230-222	404	33-26	405	218-212	406	64-59
407	267-263	408	117-114	409	115-113	410	280-279						

4 Conclusions

This was a fun exercise!

In the answer to exercise 180 in [1], several more questions are asked. Those will have to wait for another day (or perhaps the next version of this note).

References

[1] Donald Knuth, The Art of Computer Programming, Pre-fascicle 7a, Constraint satisfaction (Revision -73, 15 March 2021). Retrieved from `https://www-cs-faculty.stanford.edu/~knuth/fasc7a.ps.gz`.

[2] Donald Knuth, program downloaded from `https://cs.stanford.edu/~knuth/programs/graceful-digraphs-canonical.w`

[3] Brendan McKay, House of Graphs, digraphs up to 6 vertices. Retrieved from `http://users.cecs.anu.edu.au/~bdm/data/dig6.d6`

[4] Filip Stappers, `http://archive.org/details/diadj6grace_data`

[5] McKay, B.D. and Piperno, A., Practical Graph Isomorphism, II, Journal of Symbolic Computation, 60 (2014), pp. 94-112, `https://doi.org/10.1016/j.jsc.2013.09.003`